Results in Mathematics
Vol. 22 (1992)

0378-6218/92/020470-19$1.50+0.20/0

On solutions to coupled multiparameter nonlinear Sturm-Liouville boundary value problems whose state components have a specified nodal structure

by

Robert Stephen Cantrell *

Abstract. In preceding articles ([3] and [5]), we began an examination of the structure of the solution set to the two-parameter system

$$(*)\quad \begin{aligned} -(p_1(x)u')' + q_1(x)u &= \lambda u + f(u,v)u \\ -(p_2(x)v')' + q_2(x)v &= \mu v + g(u,v)v \end{aligned} \quad \text{in } (a,b)$$

$$u(a) = u(b) = 0 = v(a) = v(b).$$

In this article, we treat the case left uncovered in our previous analysis; namely, we assume $f(s,0) = 0$ and $g(0,t) = 0$ for all s, $t \in \mathbb{R}$. In this situation, solutions to $(*)$ of the form $(\lambda, \mu, u, 0)$ or $(\lambda, \mu, 0, v)$ lie in linear subspaces of $\mathbb{R}^2 \times (C_0^1[a,b])^2$. As such, they are neither locally expressable as functions of (λ, μ) nor are à **priori** bounded in terms of (λ, μ), as was crucial to the analysis in [3] and [5]. Nevertheless, we demonstrate that solutions to $(*)$ of the form (λ, μ, u, v) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b), where n and m are positive integers, arise as global secondary bifurcations from solutions of the form $(\lambda, \mu, u, 0)$ with u having $n-1$ simple zeros in (a,b) and from solutions of the form (λ, μ, o, v) with v having $m-1$ simple zeros in (a,b). Moreover, we establish that solutions to $(*)$ of the form (λ, μ, u, v) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b) serve as a link between solutions of the form $(\lambda, \mu, u, 0)$ with u having $n-1$ simple zeros in (a,b) and solutions of the form $(\lambda, \mu, 0, v)$ with v having $m-1$ simple zeros in (a,b). The analysis in this article when combined with that in [3] and [5] provides a fairly comprehensive examination of the structure of the solution set to $(*)$.

* The research for this article was supported in part by NSF grants DMS8802346 and DMS9002943.

1. Introduction.

Coupled systems of the form

$$-(p_iu_i')' + q_iu_i = \lambda_iu_i + f_i(u_i, .., u_n)u_i \quad \text{in } (a,b) \tag{1.1}$$

$$u_i(a) = 0 = u_i(b),$$

where λ_i is a real parameter and $i = 1, 2, ..., n$, have the property that if for instance $u_{m+1} = \cdots = u_n = 0$, where $1 \le m < n$ and $(u_1, ..., u_m)$ solves

$$-(p_iu_i') + q_iu_i = \lambda_iu_i + f_i(u_i, \cdots, u_m, 0, \cdots, 0)u_i \quad \text{in } (a,b) \tag{1.2}$$

$$u_i(a) = 0 = u_i(b)$$

for some λ_i^0, $i = 1, ..., m$, then $(u_1, \cdots, u_m, 0, \cdots, 0)$ solves (1.1) for $(\lambda_1^0, \cdots, \lambda_m^0, \lambda_{m+1}, \cdots, \lambda_n)$, with $\lambda_{m+1}, ..., \lambda_n$ arbitrary real parameters. Consequently the solution set to (1.2) can be viewed as being embedded in the solution set to (1.1) in a quite natural way. Since this phenomenon obtains if any proper subcollection of $\{u_1, ..., u_n\}$ is chosen and its members set to the zero function on $[a,b]$, (1.1) is a ready and natural systems extension of the nonlinear Sturm-Liouville boundary value problem

$$-(pu')' + qu = \lambda u + f(u)u \quad \text{in } (a,b) \tag{1.3}$$

$$u(a) = 0 = u(b). \tag{1.4}$$

Nontrivial solutions to (1.3)-(1.4) have only simple zeros. That such is the case follows from the uniqueness of solutions to initial value problems for (1.3). Consequently, the number of internal zeros of solutions to (1.3)-(1.4) can be used as a regime for classification. Such is the basis of the celebrated examination of the global structure of the solution set to (1.3)-(1.4), due to Crandall and Rabinowitz [6], among others. In particular, it is known, provided $f(0) = 0$, that there is a sequence of real numbers $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n)} < \cdots$, with $\lambda^{(n)} \to \infty$ as $n \to \infty$, such that the linear boundary value problem

$$-(pw')' + qw = \lambda^{(n)}w \quad \text{in } (a,b) \tag{1.5}$$

$$w(a) = 0 = w(b)$$

admits a solution w having $n-1$ simple zeros in (a,b) and that nontrivial solutions to (1.3)-(1.4) having $n-1$ simple zeros in (a,b) emanate (in say $\mathbb{R} \times C_0^1[a,b]$) from $(\lambda^{(n)}, 0)$. Moreover, such solutions join $(\lambda^{(n)}, 0)$ to form a one-dimensional continuum C_n of solutions to (1.3)-(1.4) which is unbounded in $\mathbb{R} \times C_0^1[a,b]$.

It is evident from the observations regarding (1.3) that if

$$-(p_iu_i')' + q_iu_i = \lambda_iu_i + f_i(u_1, .., u_n)u_i \tag{1.6}$$

in (a,b) and $u_i \not\equiv 0$, then u_i has only simple zeros in (a,b). It follows from [6] and our initial observations that there exist n-dimensional continua (subsets of $\mathbb{R}^n \times (C_0^1[a,b])^n$) of solutions to (1.1) emanating from the trivial solutions $(\lambda_1, \lambda_2, ..., \lambda_n, 0, ..., 0)$ and characterized as follows. Namely, for any such continuum $\mathcal{C}$, there is an $i \in \{1,2,...,n\}$ and a $k \in \mathbf{Z}_+$ so that if $(\lambda_1, ..., \lambda_n, u_1, ..., u_n) \in \mathcal{C}$ and $(u_1, ..., u_n) \not\equiv (0, ..., 0)$, then u_i has $k-1$ simple zeros in (a,b), $u_j \equiv 0$ for $j \neq i$, (λ_i, u_i) solves (1.6) with $u_j \equiv 0$ for $j \neq i$, and there is no constraint on λ_j, $j \neq i$. Moreover, the choice of i and k is arbitrary.

Some obvious questions arise. Do n-dimensional continua of solutions to (1.1) with more than one nontrivial state component exist? Can the number of nontrivial state components and the nodal structure of the nontrivial state components be specified at will? Does a continuum of solutions to (1.1) with ℓ nontrivial state components arise as a bifurcation from continua of solutions to (1.1) with $\ell-1$ nontrivial state components? Can two continua of solutions to (1.1) each having $\ell-1$ nontrivial state components with $\ell-2$ of them in common be linked together via a continuum of solutions to (1.1) having ℓ nontrivial state components?

We have begun an investigation of these questions in [3] and [5], dealing with the natural first case, namely $n=2$, considering the system

$$\text{(1.7)}\qquad \begin{aligned} -(p_1u')' + q_1u &= \lambda u + f(u,v)u \\ -(p_2v')' + q_2v &= \mu v + g(u,v)v \end{aligned} \quad \text{in } (a,b)$$
$$u(a) = u(b) = 0 = v(a) = v(b),$$

with $f(0,0) = 0 = g(0,0)$. In [3], we demonstrated fairly general conditions for the existence of 2-dimensional continua of solutions to (1.7) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b). Specifically, we showed that it suffices for the continuum $\mathcal{C}_n$ (in $\mathbb{R} \times C_0^1[a,b]$) of solutions to

$$-(p_1u')' + q_1u = \lambda u + f(u,0)u$$
$$u(a) = 0 = u(b)$$

emerging in $\mathbb{R} \times C_0^1[a,b]$ from $(\lambda^{(n)}, 0)$, where $\lambda^{(n)}$ is as in (1.5) to be such that the projection of $\mathcal{C}_n \cap B((\lambda^{(n)}, 0), \varepsilon)$ into $\mathbb{R}$ is not merely the singleton $\{\lambda^{(n)}\}$ for any $\varepsilon > 0$. (Here $B((\lambda^{(n)}, 0), \varepsilon)$ denotes the ball about $(\lambda^{(n)}, 0)$ of radius ε in $\mathbb{R} \times C_0^1[a,b]$.) In this case, 2-dimensional continua of solutions to (1.7) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b) emanate from $\tilde{\mathcal{C}}_n$, where $\tilde{\mathcal{C}}_n = \{(\lambda, \mu, u, 0) : (\lambda, u) \in \mathcal{C}_n\}$. In [5], we placed more specific assumptions on f and g. Namely, we assumed conditions on f and g which would model competition, predation, or mutualism in case u and v were positive. So doing, we gave a fairly broad range of examples for which continua of solutions to (1.7) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b) exist. Moreover, we demonstrated that such continua link together continua of solutions to (1.7) with u having $n-1$ simple zeros in (a,b) and v identically zero with continua of

solutions to (1.7) with u identically zero and v having $m-1$ simple zeros in (a,b). In the process, we obtained information on the range of parameters (λ, μ) for which (1.7) admits solutions with state components (u,v) of the (n,m) nodal type in a number of cases. See [5] for more details.

However, the results of [3] and [5] depended on the abstract global secondary bifurcation results of [2]. These results require that $\mathcal{C}_n$ contain a "patch" of the form $\{(\lambda, u(\lambda)) : \lambda \in I\}$, where I is an open interval and $\lambda \to u(\lambda)$ is continuous, in order to consider $\{(\lambda, \mu, u(\lambda), 0) : \lambda \in I, \mu \in \mathbb{R}\}$ as the "trivial solutions" to (1.7). Such is not possible if $f(u,0) = 0$ for all u, for then $\mathcal{C}_n = \{(\lambda^{(n)}, \alpha u_n) : \alpha \in \mathbb{R}\}$, where u_n is the unique solution to (1.5) with $\lambda = \lambda^{(n)}$ such that u_n has $n-1$ simple zeros in (a,b), $u_n'(a) > 0$, and that $\int_a^b u_n^2 = 1$. Consequently, a new approach is needed in order to establish that continua of solutions to (1.7) with state components of (n,m) nodal type emanate from $\tilde{\mathcal{C}}_n$ in this case. Moreover, in [5] the linking of solutions to (1.7) having state components of the $(n,0)$ nodal type to solutions to (1.7) having state components of the $(0,m)$ nodal type via solutions of the (n,m) nodal type made crucial use of *à priori* estimates on (u,v) available because of the presence of "self-regulation" terms in (1.7). When $f(u,0) = 0$ for all u and $g(0,v) = 0$ for all v, obtaining appropriate *à priori* estimates is far less likely, and again a new approach is called for in this part of the problem, as well.

Our purpose in this article is to demonstrate that nevertheless such bifurcations do in fact occur, and, moreover, that continua of solutions to (1.7) with state components of (n,m) nodal type do in fact link $\{(\lambda^{(n)}, \mu, \alpha u_n, 0) : \alpha \in \mathbb{R}, \mu \in \mathbb{R}\}$ to $\{(\lambda, \mu^{(m)}, 0, \beta v_m) :\in \mathbb{R}, \lambda \in \mathbb{R}\}$, where v_m is the unique solution of

$$-(p_2v_m')' + q_2v_m = \mu^{(m)}v_m \qquad \text{in } (a,b)$$
$$v_m(a) = 0 = v_m(b)$$

such that v_m has $m-1$ simple zeros in (a,b), $v_m'(a) > 0$, and $\int_a^b v_m^2 = 1$. In so doing, we make the additional assumption that $f(u,v) \equiv f(v)$ and that $g(u,v) \equiv g(u)$. The results of the article still obtain in the more general case. However, the extra assumptions make for a cleaner presentation, and the adaptations needed to generalize are evident.

The remainder of the article is structured as follows. The necessary preliminary results are given in Section 2. In Section 3, we establish that continua of solutions to (1.7) of (n,m) nodal type emanate from $\{(\lambda^{(n)}, \mu, \alpha u_n, 0) : \alpha \in \mathbb{R}, \mu \in \mathbb{R}\}$ and from $\{(\lambda, \mu^{(m)}, 0, \beta v_m) : \beta \in \mathbb{R}, \lambda \in \mathbb{R}\}$. We conclude the article in Section 4 by demonstrating the linkage of $\{(\lambda^{(n)}, \mu, \alpha u_n, 0) : \alpha \in \mathbb{R}, \mu \in \mathbb{R}\}$ and $\{(\lambda, \mu^{(m)}, 0, \beta v_m) : \beta \in \mathbb{R}, \lambda \in \mathbb{R}\}$ via a continuum of solutions to (1.7) having state components of (n,m) nodal type. The results of this article, when combined with those of [3] and [5], provide a fairly comprehensive analysis of the solution structure to (1.1) in the case when $n=2$. Moreover, they strongly suggest a similar structure to the solution set of (1.1) for general n.

2. The set-up.

Consider the system

$$\text{(2.1)}\qquad \begin{aligned} L_1u &= \lambda u + f(v)u \\ L_2v &= \mu v + g(u)v \end{aligned} \quad \text{in } (a,b), \qquad u(a) = 0 = u(b), \quad v(a) = 0 = v(b),$$

where for $i = 1, 2$, L_i denotes the differential operator given by

$$L_i z = -(p_i z')' + q_i z$$

with positive coefficients $p_i \in C^1[a,b]$ and $q_i \in C[a,b]$. The functions f, g are assumed to lie in $C^2(\mathbb{R})$ with $f(0) = 0 = g(0)$. Under these conditions, (2.1) is equivalent to the system

$$\text{(2.2)}\qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda A_1 & 0 \\ 0 & \mu A_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} A_1[f(v)u] \\ A_2[g(u)v] \end{pmatrix},$$

where for $i = 1, 2$, A_i is the inverse of the differential operator L_i (L_i subject to zero Dirichlet boundary conditions). A_i is a compact operator on $C_0^1[a,b]$. Consequently, (2.1) may be regarded as an operator equation in the Banach space $\mathbb{R}^2 \times (C_0^1[a,b])^2$, where the norms $\|||\cdot|||\|$ in $\mathbb{R}^2 \times (C_0^1[a,b])^2$ and $|||\cdot|||$ in $(C_0^1[a,b])^2$ satisfy

$$\|||(\lambda,\mu,u,v)|||\| = |\lambda| + |\mu| + |||(u,v)||| = |\lambda| + |\mu| + \|u\| + \|v\|,$$

and $\|\cdot\|$ is the usual norm in the Banach space $C^1[a,b]$. Then the right hand side of (2.2) can be expressed as

$$\text{(2.3)}\qquad G\left(\lambda,\mu,\begin{bmatrix} u \\ v \end{bmatrix}\right) = L(\lambda,\mu)\begin{bmatrix} u \\ v \end{bmatrix} + H\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

with $L(\lambda,\mu) = \begin{pmatrix} \lambda A_1 & 0 \\ 0 & \mu A_2 \end{pmatrix}$ compact linear, $H\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$ completely continuous and satisfying

$$\text{(2.4)}\qquad \lim_{|||(u,v)|||\to 0} \left(\frac{H\begin{bmatrix} u \\ v \end{bmatrix}}{|||(u,v)|||} \right) = 0$$

in $(C_0^1[a,b])^2$. It follows from (2.4) that bifurcation from the trivial solutions $(\lambda,\mu,0,0)$ to (2.1) at a point $(\bar\lambda,\bar\mu,0,0)$ is possible only when the kernel $N(I - L(\bar\lambda,\bar\mu)) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

where $I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ is the identity mapping on $(C_0^1[a,b])^2$. Suppose $\begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in N(I - L(\bar\lambda,\bar\mu))$. Then for $(\lambda,\mu) = (\bar\lambda,\bar\mu)$

$$\text{(2.5)}\qquad L_1u = \lambda u \quad \text{in } (a,b), \qquad u(a) = 0 = u(b)$$

and

$$\text{(2.6)}\qquad L_2v = \mu v \quad \text{in } (a,b), \qquad v(a) = 0 = v(b)$$

with at least one of u and v not identically zero. As is well known, the solution set to (2.5) can be expressed as

$$\{(\lambda_n, \alpha u_n) : \alpha \in \mathbb{R}, n = 1,2,3,\ldots\} \cup \{(\lambda,0) : \lambda \in \mathbb{R}\}$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots$ and $\lambda_n \to +\infty$ and u_n is the unique element of $C_0^1[a,b]$ characterized by the following:

(*i*) (λ_n, u_n) solves (2.5);

(*ii*) $u_n'(a) > 0$;

(*iii*) u_n has $n-1$ zeros in (a,b), each of which is simple;

(*iv*) $\int_a^b u_n^2\,dx = 1$.

Analogously, the solution set to (2.6) can be expressed as $\{(\mu_m, \beta v_m) : \beta \in \mathbb{R}, m = 1,2,3,\ldots\} \cup \{(\mu,0) : \mu \in \mathbb{R}\}$. Consequently, (2.1) has the two dimensional continua of solutions $\{(\lambda_n,\mu,\alpha u_n,0) : \alpha \in \mathbb{R}, \mu \in \mathbb{R}, n \in \mathbf{Z}_+\}$ and $\{(\lambda,\mu_m,0,\beta v_m) : \beta \in \mathbb{R}, \lambda \in \mathbb{R}, m \in \mathbf{Z}_+\}$ which are naturally regarded as primary bifurcations from the trivial solutions $(\lambda,\mu,0,0)$. We denote by $C_{n,0,+}$ the subcontinuum $\{(\lambda_n,\mu,\alpha u_n,0) : \alpha \geq 0, \mu \in \mathbb{R}\}$ and by $C_{n,0,-}$ the subcontinuum $\{(\lambda_n,\mu,\alpha u_n,0) : \alpha \leq 0, \mu \in \mathbb{R}\}$. We also make the evident analogous designations $C_{0,m,+}$ and $C_{0,m,-}$. If $\mu \neq \mu_m$ for any $m \in \mathbf{Z}_+$, it follows from linear Sturm-Liouville theory that $dim(\bigcup_{r\geq 1} N((I - L(\lambda_n,\mu))^r)) = 1$ for any $n \in \mathbf{Z}_+$. Likewise $dim(\bigcup_{r\geq 1} N((I - L(\lambda,\mu_m))^r)) = 1$ for any $m \in \mathbf{Z}_+$ if $\lambda \neq \lambda_n$ for any $n \in \mathbf{Z}_+$. Then the Crandall-Rabinowitz [6] or Alexander-Antman [1] Constructive Bifurcation Theorems imply that in a neighborhood of a point $(\lambda_n,\mu,0,0)$ with $\mu \neq \mu_m$ for any $m \in \mathbf{Z}_+$, the intersection of $C_{n,0,+} \cup C_{n,0,-}$ and the trivial solutions with the neighborhood is the entirety of the solution set to (2.1). An analogous statement holds in a neighborhood of $(\lambda,\mu_m,0,0)$ if $\lambda \neq \lambda_n$ for any $n \in \mathbf{Z}_+$.

3. Secondary bifurcation.

We are now in a position to seek continua $\mathcal{C}_{n,m}$ of solutions (2.1) emanating from $\mathcal{C}_{n,0,\pm}$ or $\mathcal{C}_{0,m,\pm}$, where $\mathcal{C}_{n,m}$ is to have the property that if $(\lambda,\mu,u,v)\in\mathcal{C}_{n,m}$, u has $n-1$ simple zeros in (a,b) and v has $m-1$ simple zeros in (a,b). Suppose that we have a sequence $(\lambda^i,\mu^i,u^i,v^i)$ of solutions to (2.1) with $v^i\neq 0$ and $(\lambda^i,\mu^i,u^i,v^i)$ converging to $(\lambda_n,\mu_0,\alpha_0 u_n,0)$ for some $\alpha_0\in\mathbb{R}$ and $\mu_0\in\mathbb{R}$. Then

$$v^i=\mu^i A_2 v^i + A_2(g(u^i)v^i).$$

If we let $w^i=v^i/\parallel v^i\parallel$, then $\parallel w^i\parallel=1$ and

$$w^i=\mu^i A_2 w^i + A_2(g(u^i)w^i).$$

Since $\{w^i : i\in\mathbf{Z}_+\}$ is uniformly bounded in $C_0^1[a,b]$, the compactness of A_2 implies there is a subsequence of $\{w^i : i\in\mathbf{Z}_+\}$, which we relabel if need be, so that $w^i\to\bar{w}$ in $C_0^1[a,b]$ with $\parallel\bar{w}\parallel=1$ and $\bar{w}$ satisfying

$$\bar{w}=\mu_0 A_2\bar{w}+A_2(g(\alpha_0 u_n)\bar{w}).$$

Equivalently, $\bar{w}$ satisfies

$$\begin{aligned} L_2\bar{w}-g(\alpha_0 u_n)\bar{w}&=\mu_0\bar{w} \qquad \text{in } (a,b)\\ \bar{w}(a)=0&=\bar{w}(b). \end{aligned} \tag{3.1}$$

But now for any α_0, there is a sequence $\{\mu_m^{\alpha_0,n}\}, m\in\mathbf{Z}_+$ with $\mu_m^{\alpha_0,n}<\mu_{m+1}^{\alpha_0,n}$, $\mu_m^{\alpha_0,n}\to+\infty$ as $m\to+\infty$, and $\mu_1^{\alpha_0,n}>-\parallel g(\alpha_0 u_n)\parallel_\infty$ so that (3.1) has a nontrivial solution precisely as $\mu_0=\mu_{m,}^{\alpha_0,n}, m\in\mathbf{Z}_+$. Moreover, for fixed n and m, $\mu_m^{0,n}=\mu_m$ and $\mu_m^{\alpha,n}$ depends smoothly on α, while the eigenfunction for (3.1) corresponding to $\mu_m^{\alpha,n}$ has $m-1$ simple zeros in (a,b). Consequently, we can expect secondary bifurcation from $\mathcal{C}_{n,0,\pm}$ to $\mathcal{C}_{n,m}$ only along the arc $\{(\lambda_n,\mu_m^{\alpha,n},\alpha u_n,0):\alpha\in\mathbb{R}\}$.

As noted in the introduction, to realize this secondary bifurcation, we must proceed along lines somewhat different from those employed in [3]. To this end, consider (2.1). For $u\in C_0^1[a,b]$, express u uniquely as $u=\alpha\bar{u}+w$, where $\bar{u}=u_n$ and $\int_a^b\bar{u}w=0$. Write λ as $\lambda=\lambda_n+\sigma$. Using the regularity properties of f and g, we see that $(\lambda_n+\sigma,\mu,\alpha\bar{u}+w,v)$ solves (2.1) if and only if

$$\begin{aligned} L_1 w=\lambda_n w &\ +\sigma\alpha\bar{u}+\alpha f'(0)\bar{u}v+\sigma w+f'(0)vw\\ &\ +\alpha\tilde{f}(v)\bar{u}+\tilde{f}(v)w \qquad\qquad \text{in } (a,b)\\ L_2 v=\mu v &\ +g(\alpha\bar{u})v+g'(\alpha\bar{u})wv+\tilde{g}_\alpha(w)v \end{aligned} \tag{3.2}$$

where $\lim_{s\to 0}\frac{\tilde{f}(s)}{s}=0$ and $\lim_{s\to 0}\frac{\tilde{g}_\alpha(s)}{s}$ uniformly for α contained in bounded intervals. There are three unknowns in (3.2): $w\in W=\{z\in C_0^1[a,b]:\int_a^b\bar{u}z=0\}, v\in C_0^1[a,b]$, and $\sigma\in\mathbb{R}$. We need to express (3.2) in the form

$$\vec{z}=N(\alpha,\mu,\vec{z}) \tag{3.3}$$

where $\vec{z}=\begin{pmatrix} w\\ v\\ \sigma\end{pmatrix}$ and $N:\mathbb{R}^2\times(W\times C_0^1[a,b]\times\mathbb{R})\to W\times C_0^1[a,b]\times\mathbb{R}$ is completely continuous in order to have the global bifurcation machinery of Alexander and Antman [1] at our disposal. But in order to obtain (3.3), we must have an equation for σ which is compatible with the equations in (3.2). The net result will be to excise one dimension from $(C_0^1[a,b])^2$ in (2.1) via $\alpha\bar{u}$ and compensate via σ, thinking of α and μ as the parameters of the problem. We get the necessary equation by multiplying the first equation in (3.2) by $\bar{u}$ and then integrating by parts. So doing, we obtain

$$0=\sigma\alpha+\alpha\int_a^b f(v)\bar{u}^2+\int_a^b\bar{u}f(v)w. \tag{3.4}$$

Taking the regularity properties of f into account, (3.4) is equivalent to

$$\sigma=-\int_a^b f'(0)v\bar{u}^2-\int_a^b\tilde{f}(v)\bar{u}^2-\frac{1}{\alpha}\int_a^b\bar{u}f(v)w \tag{3.5}$$

if $\alpha\neq 0$.

Combining (3.2) and (3.5), we see that (2.1) is equivalent to

$$\begin{aligned} \begin{pmatrix} w\\ v\\ \sigma\end{pmatrix}&=\begin{pmatrix} \lambda_n A_1 & \alpha f'(0)A_1(\bar{u}\cdot) & \alpha A_1(\bar{u})\\ 0 & \mu A_2+A_2[g(\alpha\bar{u})\cdot] & 0\\ 0 & -f'(0)\int_a^b\bar{u}^2(\cdot) & 0\end{pmatrix}\begin{pmatrix} w\\ v\\ \sigma\end{pmatrix}\\ &+\begin{pmatrix} \sigma A_1 w\ +f'(0)A_1(wv)+\alpha A_1(\tilde{f}(v)\bar{u})+A_1(\tilde{f}(v)w)\\ A_2 g'(\alpha\bar{u})wv+A_2\tilde{g}_\alpha(w)v\\ -\int_a^b\tilde{f}(v)\bar{u}^2-\frac{1}{\alpha}\int_a^b\bar{u}f(v)w\end{pmatrix} \end{aligned} \tag{3.6}$$

provided $\alpha\neq 0$. However, (3.6) is not yet of the form (3.3), since the right hand side of (3.6) maps $\mathbb{R}^2\times(W\times C_0^1[a,b]\times\mathbb{R})$ into $(C_0^1[a,b])^2\times\mathbb{R}$, not $W\times C_0^1[a,b]\times\mathbb{R}$ as needed. This difficulty is easily circumvented by letting P denote the projection from $C_0^1[a,b]$ to W given by $P\phi=\phi-(\int_a^b\phi\bar{u})\bar{u}$ and replacing (3.6) with the system

$$(3.7)\qquad \begin{pmatrix} w \\ v \\ \sigma \end{pmatrix} = \begin{pmatrix} \lambda_n A_1 & \alpha f'(0)A_1P(\bar{u}\cdot) & 0 \\ 0 & \mu A_2 + A_2[g(\alpha\bar{u})\cdot] & 0 \\ 0 & -f'(0)\int_a^b \bar{u}^2(\cdot) & 0 \end{pmatrix} \begin{pmatrix} w \\ v \\ \sigma \end{pmatrix}$$

$$+ \begin{pmatrix} \sigma A_1 w \quad +f'(0)A_1P(wv) + \alpha A_1P(\tilde{f}(v)\bar{u}) + A_1P(\tilde{f}(v)w) \\ A_2 g'(\alpha\bar{u})wv + A_2\tilde{g}_\alpha(w)v \\ -\int_a^b \tilde{f}(v)\bar{u}^2 - \frac{1}{\alpha}\int_a^b \bar{u}f(v)w \end{pmatrix}.$$

It is a simple matter to check that the solution sets to (3.6) and (3.7) are identical and that (3.7) is of the form (3.3).

We may now define $F : ((0,\infty)\times I\!R)\times(W\times C_0^1[a,b]\times I\!R)\to W\times C_0^1[a,b]\times I\!R$ via the right hand side of (3.7). Then $F(\alpha,\mu,\vec{z}) = L(\alpha,\mu)\,\vec{z} + H(\alpha,\mu,\vec{z})$ where

$$\vec{z} = \begin{pmatrix} w \\ v \\ \sigma \end{pmatrix}, L(\alpha,\mu) = \begin{pmatrix} \lambda_n A_1 & \alpha f'(0)A_1P(\bar{u}\cdot) & 0 \\ 0 & \mu A_2 + A_2[g(\alpha\bar{u})\cdot] & 0 \\ 0 & -f'(0)\int_a^b \bar{u}^2(\cdot) & 0 \end{pmatrix}$$

is compact linear on $W\times C_0^1[a,b]\times I\!R$, and $H\left(\alpha,\mu,\begin{pmatrix} w \\ v \\ \sigma \end{pmatrix}\right)$ is completely continuous with $\lim_{|||(w,v)|||+|\sigma|\to 0} \left(\dfrac{H\left(\alpha,\mu,\begin{pmatrix} w \\ v \\ \sigma \end{pmatrix}\right)}{|||\,(w,v)\,|||+|\,\sigma\,|} \right) = 0$ uniformly for (α,μ) in compact subsets of $(0,\infty)\times I\!R$. (Note that F could just as easily be defined on $((-\infty,0)\times I\!R)\times(W\times C_0^1[a,b]\times I\!R)$.) Since bifurcation from the trivial solutions to

$$(3.8)\qquad I - F(\alpha,\mu,\cdot) = 0,$$

where I denotes the identity mapping of $W\times C_0^1[a,b]\times I\!R$, correspond to secondary bifurcations for (2.1), the Alexander-Antman Multiparameter Global Bifurcation Theorem [1] will guarantee secondary bifurcation phenomena once an appropriate change of index is obtained for (3.8); i.e., once we obtain parameters (α_1,μ_1) and (α_2,μ_2) so that the Leray-Schauder degrees $\deg_{LS}(I-F(\alpha_1,\mu_1,\cdot),B(0,\varepsilon),0)$ and $\deg_{LS}(I-F(\alpha_2,\mu_2,\cdot),B(0,\varepsilon),0)$, where $B(0,\varepsilon)$ is the ε−ball about the origin in $W\times C_0^1[a,b]\times I\!R$, are defined and unequal. From [1, Corollary 2.46], it will suffice to find (α_1,μ_1) and (α_2,μ_2) so that $\deg_{LS}(I-L(\alpha_1,\mu_1),B(0,\varepsilon),0)\neq\deg_{LS}(I-L(\alpha_2,\mu_2),B(0,\varepsilon),0)$.

To this end, suppose $\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix} = L(\alpha,\mu)\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix}$ with $\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $I-\lambda_n A_1$ is invertible on W, it must be the case that $v = \mu A_2 v + A_2[g(\alpha\bar{u})v]$ with $v\neq 0$. Consequently, $\mu = \mu_m^{\alpha,n}$ for some $m\in\mathbf{Z}_+$. Hence if $\mu\neq\mu_m^{\alpha,n}$ for any $m\in\mathbf{Z}_+$, $\deg_{LS}(I-L(\alpha,\mu),B(0,\varepsilon),0)$ is well-defined. Suppose now such is the case for some (α,μ). If we let $T:[0,1]\times W\times C_0^1[a,b]\times I\!R\to W\times C_0^1[a,b]\times I\!R$ be defined by

$$T\left(s,\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix}\right) = \begin{pmatrix} w \quad -\lambda_n A_1 w - s\alpha f'(0)A_1P(\bar{u}v) \\ v \quad -\mu_2 A_2 v - A_2[g(\alpha\bar{u})v] \\ \sigma + sf'(0)\int_a^b \bar{u}^2 v \end{pmatrix},$$

$T\left(s,\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $v=0$ and consequently that $w=0$ and $\sigma=0$.

Hence

$$\deg_{LS}(I-L(\alpha,\mu),B(0,\varepsilon),0)$$
$$=\deg_{LS}(I-K(\alpha,\mu),B(0,\varepsilon),0),$$

where

$$(I-K(\alpha,\mu))\begin{bmatrix} w \\ v \\ \sigma \end{bmatrix} = \begin{pmatrix} w-\lambda_n A_1 w \\ v-\mu A_2 v - A_2[g(\alpha\bar{u})v] \\ \sigma \end{pmatrix}.$$

So for such an (α,μ) $\deg_{LS}(I-L(\alpha,\mu),B(0,\varepsilon),0) = \deg_{LS}(I-\lambda_n A_1,B_1(0,\varepsilon),0)\cdot\deg_{LS}(I-\mu A_2 - A_2[g(\alpha\bar{u})\cdot],B_2(0,\varepsilon),0)$, where $B_1(0,\varepsilon)$ is the ε−ball about the origin in W and $B_2(0,\varepsilon)$ is the ε−ball about the origin in $C_0^1[a,b]$. We may now state the main result of this section.

THEOREM 3.1. *There is a connected set $C_{n,m}$ of solutions to (2.1) which is of dimension ≥ 2 at each point and such that $(\lambda,\mu,u,v)\in C_{n,m}$ implies that u has $n-1$ simple zeros in (a,b) and that v has $m-1$ simple zeros in (a,b) emanating from $C_{n,0,+}$ along the curve*

$$\{(\lambda_n,\mu_m^{\alpha,n},\alpha u_n,0):\alpha>0\}.$$

Proof: The result will follow from the Alexander-Antman Multiparameter Global Bifurcation Theorem [1] and the Crandall-Rabinowitz Constructive Bifurcation Theorem [6] applied to (3.8) provided we establish that two conditions hold; first, for $\delta = \delta(\alpha) > 0$ and sufficiently small,

$$\deg_{LS}(I - (\mu_m^{\alpha,n} + \delta)A_2 - A_2[g(\alpha\bar{u})\cdot], B_2(0,\varepsilon), 0) \neq$$

$$\deg_{LS}(I - (\mu_m^{\alpha,n} - \delta)A_2 - A_2[g(\alpha\bar{u})\cdot], B_2(0,\varepsilon), 0)$$

and second that $\dim(\bigcup_{r\geq 1} N((I - L(\alpha,\mu_m^{\alpha,n}))^r)) = 1$. For the first condition, notice that for any $\alpha > 0$, there is a $\rho = \rho(\alpha) > 0$ so that $I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2$ is invertible and $\mu_m^{\alpha,n} \neq \rho$ and a corresponding $\delta^*(\alpha) > 0$ so that $\mu_m^{\alpha,n} \pm \delta - \rho \neq \mu_k^{\alpha,n}$ and $\mu_m^{\alpha,n} \pm \delta \neq \mu_k^{\alpha,n}$ for any k provided $0 < \delta < \delta^*(\alpha)$.

For such δ,

$$\begin{aligned} & I - A_2[g(\alpha\bar{u})\cdot] - (\mu_m^{\alpha,n} \pm \delta)A_2 \\ = \;& I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2 - (\mu_m^{\alpha,n} \pm \delta - \rho)A_2 \\ = \;& [I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2][I - (\mu_m^{\alpha,n} \pm \delta - \rho)(I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2)^{-1}A_2] \end{aligned}$$

It follows that

$$\begin{aligned} & \deg_{LS}(I - A_2[g(\alpha\bar{u})\cdot] - (\mu_m^{\alpha,n} \pm \delta)A_2, B_2(0,\varepsilon), 0) \\ & = \deg_{LS}(I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2, B_2(0,\varepsilon), 0) \\ & \cdot \deg_{LS}(I - (\mu_m^{\alpha,n} \pm \delta - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2, B_2(0,\varepsilon), 0). \end{aligned}$$

Hence

$$\begin{aligned} & \deg_{LS}(I - A_2[g(\alpha\bar{u})\cdot] - (\mu_m^{\alpha,n} + \delta)A_2, B_2(0,\varepsilon), 0) \\ & \neq \deg_{LS}(I - A_2[g(\alpha\bar{u})\cdot] - (\mu_m^{\alpha,n} - \delta)A_2, B_2(0,\varepsilon), 0) \end{aligned}$$

provided

$$\begin{aligned} & \deg_{LS}(I - (\mu_m^{\alpha,n} + \delta - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2, B_2(0,\varepsilon), 0) \\ & \neq \deg_{LS}(I - (\mu_m^{\alpha,n} - \delta - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2, B_2(0,\varepsilon), 0), \end{aligned}$$

which follows if $\dim N((I - (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2)^2) = 1$.

Suppose now that $z \in N(I - (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2)$.

Then $\quad z = (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2 z.$

Hence $\quad z - A_2[g(\alpha\bar{u})z] - \rho A_2 z = \mu_m^{\alpha,n}A_2 z - \rho A_2 z,$

or equivalently

$$L_2 z - g(\alpha\bar{u})z = \mu_m^{\alpha,n} z \qquad \text{in } (a,b)$$

$$z(a) = 0 = z(b).$$

Since the eigenspaces corresponding to (3.1) are one dimensional, $z = k\bar{v}$, where $\int_a^b \bar{v}^2 = 1, \bar{v}'(a) > 0$, and $\bar{v}$ has $m-1$ simple zeros in (a,b). If now $z \in N((I - (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u}\cdot] - \rho A_2]^{-1}A_2)^2)$, $(I - (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2)z = k\bar{v}$,

hence

$$\begin{aligned} & z - A_2[g(\alpha\bar{u})z] - \rho A_2 z - (\mu_m^{\alpha,n} - \rho)A_2 z \\ & = k(\bar{v} - A_2[g(\alpha\bar{u})\bar{v}] - \rho A_2\bar{v}). \end{aligned}$$

It follows readily that

$$L_2 z - g(\alpha\bar{u})z - \mu_m^{\alpha,n} z = k(\mu_m^{\alpha,n} - \rho)\bar{v}.$$

Multiplying both sides of this equation by $\bar{v}$ and integrating by parts implies $k(\mu_m^{\alpha,n} - \rho) = 0$. Hence $k = 0$, the $\dim N((I - (\mu_m^{\alpha,n} - \rho)[I - A_2[g(\alpha\bar{u})\cdot] - \rho A_2]^{-1}A_2)^2) = 1$ and the first condition is met.

To see that the second condition is met, suppose that

$$\begin{pmatrix} I_1 - \lambda_n A_1 & -\alpha f'(0)A_1(P\bar{u}\cdot) & 0 \\ 0 & I_2 - \mu_m^{\alpha,n}A_2 - A_2[g(\alpha\bar{u})\cdot] & 0 \\ 0 & f'(0)\int_a^b \bar{u}^2(\cdot) & I_3 \end{pmatrix} \begin{pmatrix} w \\ v \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where I_1, I_2, and I_3 are the identity mappings in W, $C_0^1[a,b]$, and $\mathbb{R}$, respectively. It is immediate that

$$\begin{pmatrix} w \\ v \\ \sigma \end{pmatrix} = k \begin{pmatrix} \alpha f'(0)(I_1 - \lambda_n A_1)^{-1}A_1 P(\bar{u}\bar{v}) \\ \bar{v} \\ -f'(0)\int_a^b \bar{u}^2\bar{v} \end{pmatrix}.$$

So if $(I - L(\alpha,\mu_m^{\alpha,n}))^2 \begin{pmatrix} \omega \\ \nu \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} I-\lambda_n A_1 & -\alpha f'(0)A_1 P(\bar{u}\cdot) & 0 \\ 0 & I_2-\mu_m^{\alpha,n}A_2 - A_2[g(\alpha\bar{u})\cdot] & 0 \\ 0 & f'(0)\int_a^b \bar{u}^2(\cdot) & I_3 \end{pmatrix} \begin{pmatrix} \omega \\ \nu \\ s \end{pmatrix} = k \begin{pmatrix} \alpha f'(0)(I_1-\lambda_n A_1)^{-1}A_1 P(\bar{u}\bar{v}) \\ \bar{v} \\ -f'(0)\int_a^b \bar{u}^2\bar{v} \end{pmatrix}.$$

In particular, $\nu - \mu_m^{\alpha,n}A_2\nu - A_2[g(\alpha\bar{u})\nu] = k\bar{v}$. Then $L_2\nu - \mu_m^{\alpha,n}\nu - g(\alpha\bar{u})\nu = kL_2\bar{v}$. Multiplying by $\bar{v}$ and integrating by parts yields that

$$0 = k\int_a^b [(p_2\bar{v}')^2 + q_2\bar{v}^2]dx,$$

whence $k=0$. The second condition and hence the theorem now follow.

From the equivalence of (2.1) and (3.8) for $\alpha \neq 0$, we see that $C_{n,m}$ corresponds to a continuum $\tilde{C}_{n,m}$ of nontrivial solutions to (3.8) whose existence is guaranteed by Theorem 2.2 of [1]. Theorem 2.2 of [1] applied to (3.8) asserts that $\tilde{C}_{n,m}$ is global with respect to the trivial solutions of (3.8) in the sense of conditions (2.4)-(2.6) of Theorem 2.2 of [1]. In particular, if Γ is any smooth one-dimensional restriction of the parameters $(\alpha,\mu) \in (0,\infty)\times I\!R$ crossing $(\alpha_0, \mu_m^{\alpha_0,n})$ in such a way as to affect a change of index and also meeting $\partial((0,\infty)\times I\!R)$, the restriction of $\tilde{C}_{n,m}$ along Γ either meets the trivial solutions to (3.8) at some parameter value $(\alpha,\mu) \neq (\alpha_0,\mu_m^{\alpha_0,n})$ or meets $\partial(((0,\infty)\times I\!R)\times(W\times C_0^1[a,b]\times I\!R))$.

Suppose now that Γ in addition is such that Γ meets the curve $\{(\alpha,\mu_m^{\alpha,n}) : \alpha > 0\}$ only at $(\alpha_0,\mu_m^{\alpha_0,n})$. Then the restriction to Γ of $C_{n,m}$ persists along Γ so long as the state components $(u,v) = (\alpha\bar{u}+w, v)$ remain bounded. To see that such is the case, observe that the uniqueness theorem for solutions to initial value problems for ordinary differential equations guarantees that u and v can cease to have $n-1$ and $m-1$ simple zeros in (a,b), respectively, only by vanishing identically. If $\alpha\bar{u}+w=0$, then $\alpha=0$ and $w=0$. Since $\alpha>0$ along Γ, u must continue to have $n-1$ simple zeros in (a,b). Consequently, as Γ meets $\{(\alpha,\mu_m^{\alpha,n}) : \alpha>0\}$ only at $(\alpha_0,\mu_m^{\alpha_0,n})$, v must also continue to have $m-1$ simple zeros in (a,b), and $C_{n,m}$ persists along Γ so long as (u,v) remain bounded.

A natural question arises: does $C_{n,m}$ meet $C_{0,m,+}$ or $C_{0,m,-}$? To this end, suppose that $(\alpha^k,\mu^k,w^k,v^k,\sigma^k) \in \tilde{C}_{n,m}$ and that $(\alpha^k,\mu^k,w^k,v^k,\sigma^k)$ converges to $(0,\tilde{\mu},0,\tilde{v},\tilde{\sigma})$ with $\tilde{v}\neq 0$. Then

$$L_1(\alpha^k\bar{u}+w^k) = (\lambda_n+\sigma^k)(\alpha^k\bar{u}+w^k) + f(v^k)(\alpha^k\bar{u}+w^k)$$

$$L_2 v^k = \mu^k v^k + g(\alpha^k\bar{u}+w^k)v^k$$

in (a,b). Passing to the limit in the second equation,

$$L_2\tilde{v} = \tilde{\mu}\tilde{v}$$

in (a,b). Since v^k has $m-1$ simple zeros in (a,b) and $\tilde{v}\neq 0$, $\tilde{\mu}=\mu_m$ and hence $\tilde{v}=\beta v_m$ with $\beta\neq 0$. Then

$$L_1\frac{(\alpha^k\bar{u}+w^k)}{\|\alpha^k\bar{u}+w^k\|} = (\lambda_n+\sigma^k)\frac{(\alpha^k\bar{u}+w^k)}{\|\alpha^k\bar{u}+w^k\|} + f(v^k)\frac{(\alpha^k\bar{u}+w^k)}{\|\alpha^k\bar{u}+w^k\|}$$

implies that we may assume $\dfrac{\alpha^k\bar{u}+w^k}{\|\alpha^k\bar{u}+w^k\|}$ converges to $\tilde{u}$, with $\tilde{u}$ having $n-1$ simple zeros in (a,b) and satisfying

$$L_1\tilde{u} = (\lambda_n+\tilde{\sigma})\tilde{u} + f(\beta v_m)\tilde{u}$$

in (a,b). Consequently, $\tilde{\sigma} = \lambda_n^{\beta,m}-\lambda_n$, where $\lambda_n^{\beta,m}$ is the nth eigenvalue of

(3.9) $$L_1 z - f(\beta v_m)z = \lambda z \quad \text{in } (a,b)$$

$$z(a) = 0 = z(b).$$

Consequently, if $(\lambda_n+\sigma,\ \mu,\ \alpha\bar{u}+w,\ v)\in C_{n,m}$ approaches $C_{0,m,\pm}$, it does so at a point $(\lambda_n^{\beta,m},\ \mu_m,\ 0,\ \beta v_m)$. There is no *à priori* bound on β. Consequently, it seems unlikely that we can use global bifurcation theory to establish a link between $C_{n,m}$ and $C_{0,m,\pm}$. In order to verify that such a link does in fact occur, in the next section we will make a detailed analysis of the solution set to (2.1) in a neighborhood of $(\lambda_n,\ \mu_m,\ 0, 0)$. But first let us observe fairly general conditions on f and g so that $\lim_{\beta\to\infty}\lambda_n^{\beta,m}=+\infty$ and $\lim_{\alpha\to\infty}\mu_m^{\alpha,n}=+\infty$. (This result is of independent interest as a problem in spectral theory. It also shows in the context above that frequently *à priori* bounds are not possible on the σ and v components of a solution to (3.8) as $(\alpha,\mu)\to(0,\mu_m)$.)

PROPOSITION 3.2. *Let $\lambda_n^{\beta,m}$ be as in (3.9) and assume that $f(s)\leq -ks^2$, where k is a positive constant. Then $\lim_{\beta\to+\infty}\lambda_n^{\beta,m}=+\infty$.*

Proof: Since $\lambda_n^{\beta,m}\geq\lambda_1^{\beta,m}$ for all $n\in\mathbf{Z}_+$, it suffices to prove $\lim_{\beta\to+\infty}\lambda_1^{\beta,m}=+\infty$. Furthermore, since $-f(s)\geq ks^2$, to prove $\lim_{\beta\to+\infty}\lambda_1^{\beta,m}=+\infty$ we need only show $\lim_{\beta\to+\infty}\lambda(\beta)=+\infty$, where $\lambda(\beta)$ denotes the principal eigenvalue for

(3.10) $$L_1 z + k\beta^2 v_m^2 z = \lambda z \quad \text{in } (a,b)$$

$$z(a) = 0 = z(b).$$

The variational characterization of eigenvalues for (3.10) asserts that $\lambda(\beta) = \inf\limits_{\substack{z\in H_0^1(a,b)\\ z\not\equiv 0}} \dfrac{\int_a^b p_1(z')^2+q_1z^2+k\beta^2v_m^2z^2}{\int_a^b z^2}$. Hence $\lambda(\beta)$ is monotone nondecreasing in β, so that if $\lim_{\beta\to+\infty}\lambda(\beta)\neq+\infty$, there is an $M>0$ with $\lambda(\beta)\leq M$ for all $\beta>0$.

Let z_β be a positive eigenfunction for (3.10). Since $\int_a^b p_1(z_\beta')^2 + q_1(z_\beta)^2 + \beta^2 \int_a^b kv_m^2 z_\beta^2 = \lambda(\beta)\int_a^b z_\beta^2$, $\dfrac{\int_a^b kv_m^2 z_\beta^2}{\int_a^b z_\beta^2} \leq \dfrac{\lambda(\beta)}{\beta^2}$, and $\dfrac{\int_a^b kv_m^2 z_\beta^2}{\int_a^b z_\beta^2} \to 0$ as $\beta \to +\infty$. Now consider a subsequence $\{\beta_j\}_{j=1}^\infty$ with $\beta_j \to \infty$ and z_{β_j} satisfying $\int_a^b [p_1(z_{\beta_j}')^2 + q_1 z_{\beta_j}^2] = 1$. $\{z_{\beta_j}\}_{j=1}^\infty$ is then bounded in $H_0^1(a,b)$ and hence precompact in $C[a,b]$. We may assume $z_{\beta_j} \to \bar{z}$ in $C[a,b]$. Since $\int_a^b kv_m^2 z_{\beta_j}^2 = c_j \int_a^b z_{\beta_j}^2$ with $c_j \to 0$ as $j \to \infty$, $\int_a^b kv_m^2 z_{\beta_j}^2 \to 0$ as $j \to \infty$. So $\int_a^b kv_m^2 \bar{z}^2 = 0$. kv_m^2 is nonnegative, continuous and nonzero except for $m+1$ points in $[a,b]$. So $\bar{z} \equiv 0$. Hence $\int_a^b z_{\beta_j}^2 \to 0$ as $j \to \infty$. As

$$1 + \beta_j^2 \int_a^b kv_m^2 z_{\beta_j}^2 = \lambda(\beta_j)\int_a^b z_{\beta_j}^2,$$

we must conclude that if $\lim_{\beta\to+\infty} \lambda(\beta) \neq +\infty$, then $1 \leq 0$. Consequently, $\lim_{\beta\to+\infty} \lambda(\beta) = +\infty$.

4. Linkage.

As noted at the end of the preceding section, we seek to establish that $C_{n,m}$ meets $C_{0,m,\pm}$ via an analysis of (2.1) in a neighborhood of $(\lambda_n, \mu_m, 0, 0)$. As $\lim_{\alpha\to 0} \mu_m^{\alpha,n} = \mu_m$ and $\lim_{\beta\to 0} \lambda_n^{\beta,m} = \lambda_n$, the solution curves $\{(\lambda_n, \mu_m^{\alpha,n}, \alpha u_n, 0) : \alpha \in I\!R\}$ and $\{(\lambda_n^{\beta,m}, \mu_m, 0, \beta v_m) : \beta \in I\!R\}$ approach $(\lambda_n, \mu_m, 0, 0)$ as $\alpha \to 0$ and $\beta \to 0$. Consequently, the intersection of $C_{n,m}$ and any neighborhood of $(\lambda_n, \mu_m, 0, 0)$ is nonempty. Moreover, we know by arguments analogous to those of the preceding section that continua of solutions to (2.1) with u having $n-1$ simple zeros in (a,b) and v having $m-1$ simple zeros in (a,b) emanate from $\{(\lambda_n^{\beta,m}, \mu_m, 0, \beta v_m)\}$ for $\beta > 0$ or $\beta < 0$. Additionally, the Crandall-Rabinowitz Constructive Bifurcation Theorem [6] guarantees that such continua are the only solutions to (2.1) in a neighborhood of the curve $(\lambda_n^{\beta,m}, \mu_m, 0, \beta v_m)$, $\beta > 0$ or $\beta < 0$, not of the form $(\lambda, \mu_m, 0, \beta v_m)$. Thus the linkage we seek will be established if we know that in some neighborhood of $(\lambda_n, \mu_m, 0, 0)$ we may continue $C_{n,m}$ from $(\lambda_n, \mu_m^{\alpha,n}, \alpha u_n, 0)$ to points arbitrarily close to $(\lambda_n^{\beta,m}, \mu_m, 0, \beta v_m)$. To this end, we make a Lyapunov-Schmidt reduction for (2.2) about $(\lambda_n, \mu_m, 0, 0)$, following the treatment in [4, Section 3]. We then employ the Implicit Function Theorem and the *à priori* estimates available in a neighborhood of $(\lambda_n, \mu_m, 0, 0)$ to draw the desired conclusion.

We begin with the observation that if $\lambda^* = \lambda_n$ and $\mu^* = \mu_m$, then $(C_0[a,b])^2 = N\left(I - \begin{pmatrix} \lambda^* A_1 & 0 \\ 0 & \mu^* A_2 \end{pmatrix}\right) \oplus R\left(I - \begin{pmatrix} \lambda^* A_1 & 0 \\ 0 & \mu^* A_2 \end{pmatrix}\right)$ and $T : (C_0[a,b])^2 \to (C_0[a,b])^2$ defined by

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \lambda^* A_1 u \\ v - \mu^* A_2 v \end{pmatrix} + \begin{pmatrix} (\int_a^b u\bar{u})\bar{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (\int_a^b v\bar{v})\bar{v} \end{pmatrix},$$

with $\bar{u} = u_n$ and $\bar{v} = v_m$, is a linear homeomorphism having the property that $T\begin{pmatrix} \alpha\bar{u} \\ \beta\bar{v} \end{pmatrix} = \begin{pmatrix} \alpha\bar{u} \\ \beta\bar{v} \end{pmatrix}$. It follows as in [4] (see also [7]) that (2.2) is equivalent to

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha\bar{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta\bar{v} \end{pmatrix} + T^{-1}\left[\begin{pmatrix} \sigma & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} f(v)u \\ g(u)v \end{pmatrix}\right] \tag{4.1}$$

$$\alpha = \int_a^b u\bar{u} \tag{4.2}$$

$$\beta = \int_a^b v\bar{v} \tag{4.3}$$

with $\sigma = \lambda - \lambda^*$ and $\gamma = \mu - \mu^*$. If we denote the right hand side of (4.1) by $S_{\alpha,\beta,\sigma,\gamma}\begin{pmatrix} u \\ v \end{pmatrix}$, then $S_{\alpha,\beta,\sigma,\gamma}$ is a contraction of $\overline{B((0,0),\varepsilon)} \subseteq (C_0[a,b])^2$ into itself for $|(\alpha,\beta,\sigma,\gamma)| < \delta$ for sufficiently small $\varepsilon > 0$ and $\delta > 0$. (See [4], [7].) Let $\hat{\phi} = \hat{\phi}(\alpha,\beta,\sigma,\gamma) = \begin{pmatrix} \hat{u}(\alpha,\beta,\sigma,\gamma) \\ \hat{v}(\alpha,\beta,\sigma,\gamma) \end{pmatrix}$ denote the unique fixed point in $\overline{B((0,0),\varepsilon)}$ of $S = S_{\alpha,\beta,\sigma,\gamma}$. Then $\hat{\phi}$ is smooth in $(\alpha,\beta,\sigma,\gamma)$ and $\hat{\phi} = \lim_{n\to\infty} S^n\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $S\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha\bar{u} \\ \beta\bar{v} \end{pmatrix}$, $\hat{u}(0,0,\sigma,\gamma) = 0 = \hat{v}(0,0,\sigma,\gamma)$ for all (σ,γ). Since $T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T_1 u \\ T_2 v \end{pmatrix}$ with T_1 and T_2 linear homeomorphisms from $C_0[a,b]$ onto itself, it is easy to see that in fact $\hat{u}(0,\beta,\sigma,\gamma) = 0$ for all (β,σ,γ) and $\hat{v}(\alpha,0,\sigma,\gamma) = 0$ for all (α,σ,γ). Hence $\lim_{\alpha\to 0} \dfrac{\hat{u}(\alpha,\beta,\sigma,\gamma)}{\alpha} = \dfrac{\partial}{\partial\alpha}\hat{u}(0,\beta,\sigma,\gamma)$ and $\lim_{\beta\to 0} \dfrac{\hat{v}(\alpha,\beta,\sigma,\gamma)}{\beta} = \dfrac{\partial}{\partial\beta}\hat{v}(\alpha,0,\sigma,\gamma)$. As a consequence, $\hat{u}(\alpha,\beta,\sigma,\gamma) = \alpha\tilde{u}(\alpha,\beta,\sigma,\gamma)$ and $\hat{v}(\alpha,\beta,\sigma,\gamma) = \beta\tilde{v}(\alpha,\beta,\sigma,\gamma)$ with $\tilde{u}(\alpha,\beta,\sigma,\gamma)$ and $\tilde{v}(\alpha,\beta,\sigma,\gamma)$ uniformly bounded for $|(\alpha,\beta,\sigma,\gamma)| < \delta$.

We can obtain a more detailed expansion using the fact that $\hat{\phi} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \lim_{n\to\infty} S^n\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let us suppose that $f(v) = k_1 v + \text{h.o.t.}$ and $g(u) = k_2 u + \text{h.o.t.}$ Since then $S\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha\bar{u} + T_1^{-1}\{\sigma A_1 u + A_1(k_1 vu + \text{h.o.t.})\} \\ \beta\bar{v} + T_2^{-1}\{\gamma A_2 v + A_2(k_2 uv + \text{h.o.t.})\} \end{pmatrix}$, calculations show that $S^2\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $S^3\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ have the form

$$\begin{pmatrix} \alpha\bar{u} + \sigma\alpha\lambda_n^{-1}\bar{u} + k_1\alpha\beta T_1^{-1}A_1(\bar{u}\bar{v}) + \alpha P_1(\alpha,\beta,\sigma,\gamma) \\ \\ \beta\bar{v} + \gamma\beta\mu_m^{-1}\bar{v} + k_2\alpha\beta T_2^{-1}A_2(\bar{u}\bar{v}) + \beta P_2(\alpha,\beta,\sigma,\gamma) \end{pmatrix}, \tag{4.4}$$

where P_1 and P_2 are of order at least two in all terms. It follows that $\hat{\phi}$ must have the same form. Now (2.2) is solvable precisely when $\hat{u}(\alpha,\beta,\sigma,\gamma)$ solves (4.2) and $\hat{v}(\alpha,\beta,\sigma,\gamma)$ solves (4.3). From (4.4), these conditions for solvability of (2.2) reduce to

$$
(4.5)\qquad
\begin{aligned}
&\sigma\alpha\lambda_n^{-1} + k_1\alpha\beta\int_a^b [T_1^{-1}A_1(\bar{u}\bar{v})]\bar{u} + \alpha\int_a^b P_1(\alpha,\beta,\sigma,\gamma)\bar{u} = 0\\
&\gamma\beta\mu_m^{-1} + k_2\alpha\beta\int_a^b [T_2^{-1}A_2(\bar{u}\bar{v})]\bar{v} + \beta\int_a^b P_2(\alpha,\beta,\sigma,\gamma)\bar{v} = 0.
\end{aligned}
$$

When $\alpha \neq 0$ and $\beta \neq 0$, (4.5) is equivalent to

$$
(4.6)\qquad
\begin{aligned}
&\sigma\lambda_n^{-1} + k_1\beta\int_a^b [T_1^{-1}A_1(\bar{u}\bar{v})]\bar{u} + \int_a^b P_1(\alpha,\beta,\sigma,\gamma)\bar{u} = 0\\
&\gamma\mu_m^{-1} + k_2\alpha\int_a^b [T_2^{-1}A_2(\bar{u}\bar{v})]\bar{v} + \int_a^b P_2(\alpha,\beta,\sigma,\gamma)\bar{v} = 0.
\end{aligned}
$$

Now define $F : B((0,0,0,0),\delta) \subseteq \mathbb{R}^4 \to \mathbb{R}^2$ by

$$
F\begin{pmatrix}\alpha\\ \beta\\ \sigma\\ \gamma\end{pmatrix} = \begin{pmatrix} F_1(\alpha,\beta,\sigma,\gamma)\\ F_2(\alpha,\beta,\sigma,\gamma)\end{pmatrix}
$$

$$
= \begin{pmatrix} \sigma\lambda_n^{-1} + k_1\beta\int_a^b [T_1^{-1}A_1(\bar{u}\bar{v})]\bar{u} + \int_a^b P_1(\alpha,\beta,\sigma,\gamma)\bar{u}\\ \gamma\mu_m^{-1} + k_2\alpha\int_a^b [T_2^{-1}A_2(\bar{u}\bar{v})]\bar{v} + \int_a^b P_2(\alpha,\beta,\sigma,\gamma)\bar{v}\end{pmatrix}.
$$

Then
$$
\begin{pmatrix} \dfrac{\partial F_1}{\partial\sigma} & \dfrac{\partial F_1}{\partial\gamma}\\[2mm] \dfrac{\partial F_2}{\partial\sigma} & \dfrac{\partial F_2}{\partial\gamma}\end{pmatrix} =
$$

$$
\begin{pmatrix} \lambda_n^{-1} + \int_a^b \dfrac{\partial P_1}{\partial\sigma}(\alpha,\beta,\sigma,\gamma)\bar{u} & \int_a^b \dfrac{\partial P_1}{\partial\gamma}(\alpha,\beta,\sigma,\gamma)\bar{v}\\ \int_a^b \dfrac{\partial P_2}{\partial\sigma}(\alpha,\beta,\sigma,\gamma)\bar{u} & \mu_m^{-1} + \int_a^b \dfrac{\partial P_2}{\partial\gamma}(\alpha,\beta,\sigma,\gamma)\bar{v}\end{pmatrix}.
$$

Since $\lambda_n \neq 0$ and $\mu_m \neq 0$, it follows that for $|\,(\alpha,\beta,\sigma,\gamma)\,|$ sufficiently small, $F\begin{pmatrix}\alpha\\ \beta\\ \sigma\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ may be solved for (σ,γ) in terms of (α,β). As a consequence, the solution set to (2.2) can be expressed in the form

$$
(4.7)\quad (\lambda_n + \sigma(\alpha,\beta), \mu_m + \gamma(\alpha,\beta), \hat{u}(\alpha,\beta,\sigma(\alpha,\beta),\gamma(\alpha,\beta)), \hat{v}(\alpha,\beta,\sigma(\alpha,\beta),\gamma(\alpha,\beta)))
$$

near any solution $(\lambda_n + \sigma, \mu_m + \gamma, \hat{u}(\alpha,\beta,\sigma,\gamma), \hat{v}(\alpha,\beta,\sigma,\gamma))$ with $|\,(\alpha,\beta,\sigma,\gamma)\,|$ sufficiently small, $\alpha \neq 0$ and $\beta \neq 0$. The remarks at the beginning of this section show that $C_{n,m}$ links to $C_{0,m,\pm}$ provided we can continue solutions of the form (4.7) for $|\,(\alpha,\beta,\sigma,\gamma)\,|$ sufficiently small and $\alpha \neq 0,\ \beta \neq 0$.

Suppose now that $\{(\alpha^k,\beta^k,\sigma^k,\gamma^k)\}_{k=1}^{\infty}$ is a sequence in $B((0,0,0,0),\delta)$ such that $\alpha^k \neq 0,\ \beta^k \neq 0$,

$$
\begin{aligned}
\alpha^k &= \int_a^b \hat{u}(\alpha^k,\beta^k,\sigma^k,\gamma^k)\bar{u}\\
\beta^k &= \int_a^b \hat{v}(\alpha^k,\beta^k,\sigma^k,\gamma^k)\bar{v}
\end{aligned}
$$

and $\alpha^k \to 0$ and $\beta^k \to 0$. Let $\hat{u}_k = \hat{u}(\alpha^k,\beta^k,\sigma^k,\gamma^k)$ and $\hat{v}_k = \hat{v}(\alpha^k,\beta^k,\sigma^k,\gamma^k)$. Then $\hat{u}_k \neq 0$ and $\hat{v}_k \neq 0$ satisfy

$$
\begin{aligned}
L_1\hat{u}_k &= (\lambda_n + \sigma^k)\hat{u}_k + f(\hat{v}_k)\hat{u}_k\\
L_2\hat{v}_k &= (\mu_m + \gamma^k)\hat{v}_k + g(\hat{u}_k)\hat{v}_k
\end{aligned}
\qquad \text{in } (a,b)
$$

with $\hat{u}_k(a) = 0 = \hat{u}_k(b)$ and $\hat{v}_k(a) = 0 = \hat{v}_k(b)$. Since $\hat{u}_k = \alpha^k\tilde{u}(\alpha^k,\beta^k,\sigma^k,\gamma^k)$ and $\hat{v}_k = \beta^k\tilde{v}(\alpha^k,\beta^k,\sigma^k,\gamma^k)$, where $\tilde{u}$ and $\tilde{v}$ are uniformly bounded for $(\alpha,\beta,\sigma,\gamma) \in B((0,0,0,0),\delta)$, $\hat{u}_k \to 0$ and $\hat{v}_k \to 0$. We may now employ a compactness argument to assert that a subsequence of $\left(\dfrac{\hat{u}_k}{\|\hat{u}_k\|}, \dfrac{\hat{v}_k}{\|\hat{v}_k\|}\right)$ converges to (u^*,v^*), where $\|u^*\| = 1 = \|v^*\|$ and u^* and v^* satisfy

$$
\begin{aligned}
L_1u^* &= (\lambda_n + \bar{\sigma})u^*\\
L_2v^* &= (\mu_m + \bar{\gamma})v^*
\end{aligned}
\qquad \text{in } (a,b)
$$

with $u^*(a) = 0 = u^*(b)$, $v^*(a) = 0 = v^*(b)$ and $(0,0,\bar{\sigma},\bar{\gamma}) \in \overline{B((0,0,0,0),\delta)}$. Consequently, $(\bar{\sigma},\bar{\gamma}) = (0,0)$. So if $\rho > 0$ is given, there is a $\eta > 0$ so that if $|\,(\alpha,\beta)\,| < \eta$, $\alpha \neq 0$, $\beta \neq 0$, and

$$
\alpha = \int_a^b \hat{u}(\alpha,\beta,\sigma,\gamma)\bar{u}
$$

$$
\beta = \int_a^b \hat{v}(\alpha,\beta,\sigma,\gamma)\bar{v}
$$

for $(\alpha,\beta,\sigma,\gamma) \in B((0,0,0,0),\delta)$, then $|\,(\sigma,\gamma)\,| < \rho$.

Finally, to establish the linkage of $C_{n,m}$ to $C_{0,m,\pm}$, we proceed as follows. Choose $(\alpha_0,\beta_0,\sigma_0,\gamma_0)$ so that $\alpha_0 > 0$, $\beta_0 \neq 0$, $|\,(\alpha_0,\beta_0)\,| < \eta$, $|\,(\sigma_0,\gamma_0)\,| < \rho$ and $(\lambda_n + \sigma_0, \mu_m + \gamma_0, \hat{u}(\alpha_0,\beta_0,\sigma_0,\gamma_0), \hat{v}(\alpha_0,\beta_0,\sigma_0,\gamma_0)) \in C_{n,m}$. The preceding results guarantee that we can express $C_{n,m}$ in a neighborhood of $(\lambda_n + \sigma_0, \mu_m + \gamma_0, \hat{u}(\alpha_0,\beta_0,\sigma_0,\gamma_0), \hat{v}(\alpha_0,\beta_0,\sigma_0,\gamma_0))$ as a function of (α,β) and that this function can be continued (keeping $\alpha > 0$, $\beta \neq 0$) into arbitrarily small neighborhoods of the curve of solutions $\{(\lambda_n^{\beta,m}, \mu_m, 0, \beta v_m)\}$, for $\beta > 0$ if $\beta_0 > 0$ and for $\beta < 0$ if $\beta_0 < 0$. As previously noted, the only such solutions are those emanating from $C_{0,m,\pm}$. Consequently, $C_{n,m}$ must link to $C_{0,m,+}$ or $C_{0,m,-}$.

References

1. J.C. Alexander and S.S. Antman, Global and local behavior of bifurcating multidimensional continua of solutions for multiparameter nonlinear eigenvalue problems, Arch. Rational Mech. Anal. **76** (1981), 339-354.

2. R.S. Cantrell, Global higher bifurcations in coupled systems of nonlinear eigenvalue problems, Proc. Roy. Soc. Edinburgh Sect.A **106** (1987), 113-120.

3. R.S. Cantrell, Global preservation of nodal structure in coupled systems of nonlinear Sturm-Liouville boundary value problems, Proc. Amer. Math. Soc. **107** (1989), 633-644.

4. R.S. Cantrell, On coupled multiparameter nonlinear elliptic systems, Trans. Amer. Math. Soc. **294** (1986), 263-285.

5. R.S. Cantrell, Parameter ranges for the existence of solutions whose state components have specified nodal structure in coupled multiparameter systems of nonlinear Sturm-Liouville boundary value problems, Proc. Roy. Soc. Edinburgh Sect.A, to appear.

6. M.G. Crandall and P.H. Rabinowitz, Nonlinear Sturm-Liouville eigenvalue problems and topological degree, J. Math. Mech. **19** (1970), 1083-1102.

7. D.W. Zachmann, Multiple solutions of coupled Sturm-Liouville systems, J. Math. Anal. Appl. **54** (1976), 467-475.

Department of Mathematics and Computer Science
The University of Miami
Coral Gables, Florida 33124

Eingegangen am 26. Mai 1991